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Finite Element Analysis of Planar Microwave Networks

P. SILVESTER

***Abstract*—The port admittance matrix of a planar network is formulated in terms of certain harmonic functions related to the port voltages and the network geometry, together with the natural modes of the network with all ports shorted. The necessary harmonic functions and eigenfunctions are found using a finite element technique, for which general-purpose computer programs already exist. An advantage of the method is that the admittance matrix appears in partial-fraction form with geometric data separated from frequency, leading to inexpensive computations where recalculation at various frequencies is required.**

Introduction

PLANAR multiport microwave networks offer the designer considerable freedom as compared to stripline circuitry, not only in regard to physical size and shape, but more importantly, to such electrical characteristics as impedance level. Considerable interest has therefore arisen in their analysis and design in recent years [1].

A very comprehensive theory, leading to a partial-fraction representation of the admittance matrices of the general N-port, was given by Civalleri and Ridella [2]. In applications, the full generality of this theory is not always required; a simplified version, based on a somewhat more idealized formulation of the problem, is often entirely adequate as demonstrated by the results of Bianco and Ridella [3]. Their analysis, however, was restricted to rectangular circuits, for which certain eigenfunctions are analytically known. While interesting in pointing out certain possible network behavior patterns, restriction to rectangular plates robs the designer in large measure of precisely that flexibility promised by planar networks. An extension of their formulation or an alternative formulation not so geometrically restrictive would therefore seem desirable. An alternative approach published concurrently by Okoshi and Miyoshi [4] formulated the field problem of the planar circuit in terms of a Fredholm integral equation—similarly to Spielman [5]—which was subsequently solved by a collocation method. This approach lends itself well to computational implementation and does not involve undue geometrical constraints. A drawback of this technique, however, is that the resulting network characterization (be it a transfer matrix or an admittance matrix) is valid at only one frequency; for any other frequency, the entire integral equation analysis must be repeated.

The analysis given below is geometrically as little restricted as the method of Okoshi and Miyoshi, but the network admittance matrices which result are in partial-fraction form. Consequently, it is only necessary to solve the field problem for a given network once; there is no need for repeated analyses at different frequencies. Thus although the new method differs fundamentally from those reported earlier, it combines in one the advantages of both existing methods.

Formulation of Field Problem

For purposes of analysis, exactly the same idealizations will be employed in this paper as in previous ones [3], [4]. The planar network will be assumed to consist of a highly conductive plate placed on a dielectric substrate backed by a conductive ground plane. Both the dielectric and the ground plate are assumed infinite in extent and analysis will be carried out for the equivalent structure of two similar plates separated by an infinite dielectric sheet of double thickness, as in Fig. 1. It will be assumed that the plate lateral dimensions are very much greater than the dielectric thickness, so that the electric field may be assumed everywhere normal to the two plates, $\boldsymbol{E}=\mathbf{1}_z E_z$. That is to say, fringing fields at plate edges are ignored. The network is assumed to be fed by N-ports arranged along its periphery in such a way that no two-ports have any points in common along the periphery. Within the

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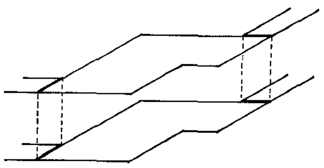

Fig. 1. Arbitrary polygonal planar network, analytically treated as two plates separated by a dielectric sheet.

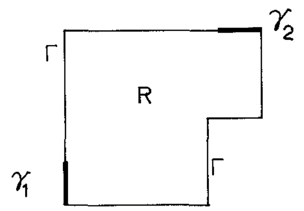

Fig. 2. The polygonal network plate of Fig. 1, showing nomenclature for analysis. Note that Γ refers to the open portion of the network boundary only.

dielectric-filled space between the plates the electric field must satisfy

$$(\nabla^2 + \Omega^2)E_z = 0 \tag{1}$$

where Ω denotes the normalized frequency:

$$\Omega = \omega\sqrt{\mu\epsilon}. \tag{2}$$

As is well known [6], the magnetic field in the dielectric must then be given by

$$\boldsymbol{H} = \frac{1}{j\omega\mu}(\mathbf{1}_z \times \nabla_{xy}E_z) \tag{3}$$

where ∇_{xy} denotes the transverse gradient operator. Above and below the planar network, there can exist no magnetic field related to the internal fields. In accordance with Maxwell's equations, there must therefore flow in the upper plate a surface current whose density is given by

$$\boldsymbol{J} = \frac{1}{j\omega\mu}\nabla_{xy}E_z. \tag{4}$$

All fields are understood to be time harmonic. Along the open plate edge Γ, the surface current density cannot have an outward normal component; hence (4) requires that along Γ

$$\frac{\partial E_z}{\partial n} = 0. \tag{5}$$

On the other hand, along the section of the periphery covered by the kth port, say γ_k, an outward current is permissible. Its value, according to (4), must be

$$I_k = \frac{1}{j\omega\mu}\int_{\gamma_k}\frac{\partial E_z}{\partial n}\,dS. \tag{6}$$

It is frequently most convenient to formulate the problem not in terms of the field $\boldsymbol{E}$, but the RF voltage v between corresponding points of the upper and lower plates. Let h be the plate spacing. Since the field has been assumed entirely z-directed, $v = hE_z$; the two formulations are entirely equivalent. If all ports are taken to be sufficiently narrow and to be fed by pure TEM mode lines, the following boundary-value problem results over the region of the upper plate, as shown in Fig. 2

$$(\nabla^2 + \Omega^2)v = 0, \qquad \text{in } R \tag{7}$$

$$\frac{\partial v}{\partial n} = 0, \qquad \text{on } \Gamma \tag{8}$$

$$v = V_k, \qquad \text{on } \gamma_k, \quad k = 1, 2, \cdots, N. \tag{9}$$

This is a mixed scalar boundary-value problem in the ordinary form and may be solved as indicated below.

Formal Solution of the Field Problem

The boundary-value problem defined by (7)–(9) comprises a homogeneous differential equation in R with inhomogeneous boundary conditions. For numerical solution, homogeneous boundary conditions and an inhomogeneous differential equation are to be preferred. The latter situation may be brought about by a simple change of variables. Let u denote a function which satisfies the boundary-value problem

$$\nabla^2 u = 0, \qquad \text{in } R \tag{10}$$

$$\frac{\partial u}{\partial n} = 0, \qquad \text{on } \Gamma \tag{11}$$

$$u = V_k, \qquad \text{on } \gamma_k, \quad k = 1, 2, \cdots, N. \tag{12}$$

Now let

$$v = u + w. \tag{13}$$

Substitution in (7)–(9) then produces

$$(\nabla^2 + \Omega^2)w = -\,\Omega^2 u, \qquad \text{in } R \tag{14}$$

$$\frac{\partial w}{\partial n} = 0, \qquad \text{on } \Gamma \tag{15}$$

$$w = 0, \qquad \text{on } \gamma_k. \tag{16}$$

In this way, solution of the boundary-value problem (7)–(9) has been reduced to the solution of two other boundary-value problems. It remains to write the solutions in such a form as to exhibit clearly their dependence on the port voltages.

To begin, assume that all port voltages V_k are zero (port shorted) except for the ith, which will be assumed to have unity value:

$$V_k = 0, \qquad k \neq i$$
$$= 1, \qquad k = i. \tag{17}$$

Let the boundary-value problem (10)–(12) be solved, subject to the restriction (17), $i = 1, 2, 3, \cdots, N$. Let Ψ_i denote the ith solution. For any possible set of port voltages $\{V_k\}$, the solution of (10)–(12) is exactly

$$u = \sum_{i=1}^{N} V_i\psi_i. \tag{18}$$

Next, consider the eigenvalue problem defined by

$$(\nabla^2 + \Omega^2)w = 0, \qquad \text{in } R \tag{19}$$

and the boundary conditions (15)–(16). Let the eigenvalues and eigenfunctions which satisfy it be Ω_i and ϕ_i, $i = 1, 2, \cdots$. These eigenfunctions may be used as basis functions for Galerkin projective solution of (14)–(16). Let w be expanded as

$$w = \sum_{k=1}^{\infty} W_k \phi_k \tag{20}$$

where the W_k are coefficients. Substitution of (18)–(20) into (14) then produces

$$\sum_{k=1}^{\infty} W_k(\nabla^2 + \Omega^2)\phi_k = -\Omega^2 \sum_{i=1}^{N} V_i \psi_i \tag{21}$$

which may be written, since ϕ_k satisfies the Helmholtz eigenvalue problem, as

$$\sum_{k=1}^{\infty} W_k(\Omega^2 - \Omega_k^2)\phi_k = -\Omega^2 \sum_{i=1}^{N} V_i \psi_i. \tag{22}$$

Let (22) be multiplied by ϕ_m and integrated over the network region R. Since the functions ϕ_k constitute an orthogonal family, only the member $k=m$ of the left-hand summation survives. Therefore

$$W_k = \frac{\Omega^2}{\Omega_k^2 - \Omega^2} \frac{1}{\int_R \phi_k^2 \, dR} \sum_{i=1}^{N} V_i \int_R \psi_i \phi_k \, dR. \tag{23}$$

Since the port voltages are assumed to be known, the coefficients W_k are readily calculable at any frequency. The complete field solution is therefore

$$v = \sum_{i=1}^{N} V_i \left\{ \psi_i + \sum_{k=1}^{\infty} \frac{\Omega^2}{\Omega_k^2 - \Omega^2} \frac{\int_R \psi_i \phi_k \, dR}{\int_R \phi_k^2 \, dR} \phi_k \right\}. \tag{24}$$

It should be noted that the boundary-value problems which define functions ψ_i and ϕ_k are of a form which can be solved by existing computer programs with good accuracy [7]. For purposes of the above discussion, therefore, they may be regarded as known once the boundary-value problems have been stated.

Admittance Matrix of the General N-Port

Rarely is the actual RF voltage distribution over the network desired; usually, the designer wishes only to have a terminal description, preferably as an impedance or admittance matrix. The latter, as will be seen, is readily available.

In terms of the RF voltage, the outward current at the mth port (6) reads

$$I_m = \frac{1}{j\omega\mu h} \int_{\gamma_m} \frac{\partial v}{\partial n} \, ds. \tag{25}$$

Substitution of (24), followed by some algebraic manipulation, then produces

$$I_m = -j \frac{\sqrt{\epsilon/\mu}}{h} \sum_{i=1}^{N} \left(\sum_{k=0}^{\infty} \frac{\Omega}{\Omega_k^2 - \Omega^2} B_{mi}^{(k)} \right) V_i \tag{26}$$

where the $N \times N$ matrices $B^{(k)}$ are defined elementwise by

$$B_{mi}^{(0)} = -\int_{\gamma_m} \frac{\partial \psi_i}{\partial n} \, ds \tag{27}$$

with $\Omega_0 = 0$ understood and

$$B_{mi}^{(k)} = \frac{\int_R \psi_i \phi_k \, dR}{\int_R \phi_k^2 \, dR} \int_{\gamma_m} \frac{\partial \phi_k}{\partial n} \, ds, \qquad k = 1, 2, \cdots, N. \tag{28}$$

The currents in (26) are taken as outward flowing in keeping with the outward normal convention of field theory. Taking network currents to have inward-directed references, the admittance matrix of the N-port thus reads

$$Y = j \frac{\sqrt{\epsilon/\mu}}{h} \sum_{k=0}^{\infty} \frac{\Omega}{\Omega_k^2 - \Omega^2} B^{(k)}. \tag{29}$$

This expression, it may be noted, is in partial-fraction or Foster canonic form, with the eigenfrequencies of the network cavity corresponding to the admittance poles. The matrices $B^{(k)}$ have as many rows and columns as the network has ports; they are pure numerics, their elements depending solely on the geometric shape and size of the network. Consequently, once the geometric matrices $B^{(k)}$ have been computed, the network admittance matrix at any frequency may be obtained to an adequate accuracy by summing the first few terms of (29)—a trivial computational task.

From the form of (29), it is clear that the admittance matrix Y cannot be symmetric at all frequencies unless each and every one of the matrices $B^{(k)}$, $k = 0, 1, \cdots$, is symmetric also. However, (27) and (28) do not exhibit any symmetry in the indices m and i. Since the planar network under consideration is passive, linear, and bilateral, symmetry must obtain. It is therefore desirable to recast the latter two equations in a form such as to prove symmetry explicitly.

The functions ψ_m are interpolative portwise, according to (12) and (17); the functions ϕ_k all have zero normal derivative along the network boundary, except at the ports. Therefore

$$\int_{\gamma_m} \frac{\partial \phi_k}{\partial n} \, ds = \int_{\partial R} \psi_m \frac{\partial \phi_k}{\partial n} \, ds \tag{30}$$

where

$$\partial R = \Gamma \cup \left(\bigcup_{i=1}^{N} \gamma_i \right)$$

is the boundary of the network. This closed line integral, taken around the network periphery, may be written in the form given by Green's theorem:

$$\oint_{\partial R} \psi_m \frac{\partial \phi_k}{\partial n} \, ds = \int_R \psi_m \nabla^2 \phi_k \, dR - \int_R \phi_k \nabla^2 \psi_m \, dR + \oint_{\partial R} \phi_k \frac{\partial \psi_m}{\partial n} \, ds. \tag{31}$$

In accordance with the definitions given for ϕ_k and ψ_m above, either ϕ_k or the normal derivative of ψ_m is zero at each and every boundary point. The rightmost integral in (31) therefore vanishes. By definition, the functions ψ_m are harmonic; therefore the middle integral on the right side of (31) vanishes. Furthermore, ϕ_k satisfies Helmholtz's equation in R, so that

(31) simplifies to

$$\oint_{\partial R} \psi_m \frac{\partial \phi_k}{\partial n} ds = -\Omega_k^2 \int_R \psi_m \phi_k \, dR. \quad (32)$$

Substitution of (30) and (32) into (29) now produces the alternative form for $B_{mi}^{(k)}$, $k=1, 2, \cdots$:

$$B_{mi}^{(k)} = -\Omega_k^2 \frac{\int_R \psi_i \phi_k \, dR \int_R \psi_m \phi_k \, dR}{\int_R \phi_k^2 \, dR} \quad (33)$$

where symmetry is obvious.

Symmetry of the matrix $B^{(0)}$ can be proved by a similar development. Equation (30) is still valid if ϕ_k is replaced by ψ_i; using Green's theorem again, one thus obtains

$$\int_{\gamma_m} \frac{\partial \psi_i}{\partial n} ds = \int_R \nabla\psi_i \cdot \nabla\psi_m \, dR + \int_R \psi_m \nabla^2 \psi_i \, dR. \quad (34)$$

The rightmost term again vanishes, since ψ_i is a harmonic function. There results, finally,

$$B_{mi}^{(0)} = -\int_R \nabla\psi_i \cdot \nabla\psi_m \, dR \quad (35)$$

again clearly symmetric.

Computational Formulation and Tests

To obtain an economic computational implementation of the above analytic development, the finite element method, using triangular elements, will be employed. Let the planar network region R be triangulated in such a way that each triangle side along the periphery of R is either entirely along one port or else touches ports at most at its end points. Let the functions ψ_i and ϕ_k be approximated by continuous piecewise polynomials on each triangle, as is usual in the finite element method [7]–[9]. The functions themselves are then representable by vectors of interpolation coefficients Ψ_i and Φ_k. It is readily shown [9] that if $\tilde{\psi}_i$ and $\tilde{\phi}_k$ are the polynomial approximations to ψ_i and ϕ_k, respectively, then

$$\int_R \tilde{\psi}_i \tilde{\phi}_k \, dR = \Psi_i' T \Phi_k \quad (36)$$

and

$$\int_R \nabla\tilde{\psi}_i \cdot \nabla\tilde{\psi}_m \, dR = \Psi_i' S \Psi_m. \quad (37)$$

Here primes denote transposition, S is the matrix of Dirichlet integrals of the interpolation functions associated with the triangulation, while T is the metric of the interpolation functions. These matrices are automatically produced by readily available Fortran subprograms [7], so that no further theory need be developed for their construction. Furthermore, the approximate harmonic functions $\tilde{\psi}_i$ and the approximate eigenfunctions $\tilde{\phi}_k$ are determined by the same general-purpose finite element programs in the form of coefficient vectors Ψ_i and Φ_k. Using (33) and (35) as the computational prescription, one thus obtains

$$B_{mi}^{(k)} = -\Omega_k^2 \frac{\Psi_m' T \Phi_k \Phi_k' T \Psi_i}{\Phi_k' T \Phi_k}, \qquad k=1, 2, \cdots, N \quad (38)$$

and

$$B_{mi}^{(0)} = -\Psi_i' S \Psi_m. \quad (39)$$

Clearly, the programming effort here is minimal, if the standard programs are used without alteration. Their use in this form is perhaps not quite so economic as might be that of a special program written particularly to produce admittance matrices of planar networks. The latter course may have much to recommend it if large numbers of networks are to be analyzed, as might be the case in an iterative synthesis procedure.

The computational procedure given by (29), (38), and (39) is quite inexpensive to execute. For K terms in the series in (29), evaluation of the admittance matrix at each frequency requires approximately $\frac{1}{2}KN^2$ accumulative (multiply–add–store) operations. For example, for a 30-term expansion describing a 4-port network, 155 operations are required or about 5 ms on a reasonably fast computer. The initial overhead of solving the necessary boundary-value problems and computing the matrices $B^{(k)}$ is approximarely $2M^3$ operations, where M is the order of the finite element matrices S and T. These timings do not depend on network shape, except as it may be reflected in the matrix order M.

An alternative procedure might be to employ (27) and (28) directly as computational algorithms. Such a course of action has the drawback of more extensive programming, but results in slightly faster execution times. The normal derivatives of polynomial approximations are clearly also polynomials, so that all differentiations and integrations can be carried out exactly (aside from roundoff error). A program to implement this approach was also written using the normal differentiation operators published recently [10] in conjunction with Newton–Cotes quadrature formulas; the latter choice is natural since the nodes along one triangle edge for the usual triangle interpolation polynomials coincide with the nodes of closed-form one-dimensional Newton–Cotes formulas. Not surprisingly, results obtained in this way show only slight differences from those obtained from (38) and (39). These are presumably attributable to differences in roundoff error incurred by the two distinct arithmetic processes.

The model problem used for program testing is a rectangular strip L units long and one length unit wide, with the ports occupying the full width of both narrow ends. Provided L is much greater and the substrate thickness much smaller than unity, such a structure constitutes a good approximation to an idealized parallel-plate TEM transmission line. The admittance matrix of the latter is easily shown to be

$$Y = jY_0 \begin{bmatrix} -\cot \Omega L & \csc \Omega L \\ \csc \Omega L & -\cot \Omega L \end{bmatrix} \quad (40)$$

where it is assumed, as usual in transmission line theory, that no lateral variations in voltage can occur. For this problem, the necessary functions are analytically determinable. It is easily verified that the eigenfunctions and eigenvalues are

$$\phi_k = \sin \frac{k\pi x}{L} \qquad \Omega_k = \frac{k\pi}{L} \quad (41)$$

while the port functions are given by

$$\psi_1 = \frac{x}{L} \quad (42)$$

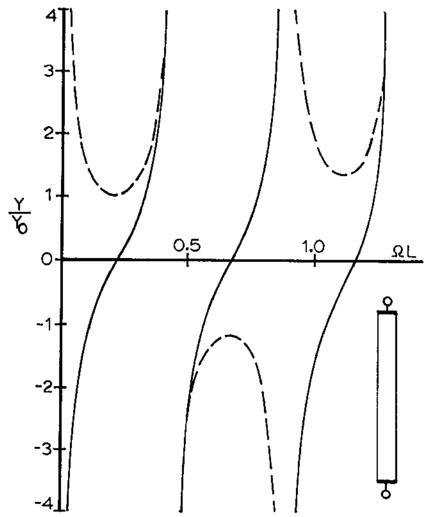

Fig. 3. Normalized admittances for the model problem. Solid line denotes Y_{11}, dashed line Y_{12}. Effect of laterally varying eigenfunctions is discernible in Y_{12} at the higher frequencies.

$$\psi_2 = 1 - \frac{x}{L} \cdot \tag{43}$$

The matrices $B^{(k)}$ are in this case easily evaluated. They are

$$B^{(0)} = \frac{1}{L}\begin{bmatrix} -1 & +1 \\ +1 & -1 \end{bmatrix} \tag{44}$$

$$B^{(k)} = \frac{2}{L}\begin{bmatrix} 1 & (-1)^k \\ (-1)^k & 1 \end{bmatrix}, \qquad k \geq 1. \tag{45}$$

Because the quantities involved are all determined as surface integrals, the corresponding numerical processes are not very roundoff sensitive. For example, Fig. 3 shows the results obtained computationally for the case $L=9$. The network was modeled by six identical right-triangular finite elements. Several computational tests were carried out, using various degrees of polynomial approximation. To produce Fig. 3, second-order polynomials were used, so that the matrices S and T were of order 21. Computations were carried out including elementary matrices $B^{(0)}$–$B^{(15)}$. Up to the third pole, Y_{11} and Y_{12} agree with (40) to four significant figures; since all calculations were carried out with a 24-bit mantissa, this level of agreement is considered entirely satisfactory. Beyond the third pole, inclusion in the computational analysis of laterally varying eigenfunctions ϕ_k precludes detailed agreement with (40)—indeed, it is the approximations inherent in (40), and not the approximations of the computational analysis which break down first. This view is substantiated by comparison of the analytically obtained matrices $B^{(k)}$, with corresponding ones obtained from the finite element analysis.

A variety of different networks has been analyzed using the new programs. For networks which cannot be treated analytically, no absolute error assessment can be given. However, the only mathematical approximations involved in the procedure are those inherent in finite element solution to find the basis functions: truncation of the series expansion and approximation of all functions by interpolation polynomials. It is therefore reasonable to expect an overall accuracy consistent with the accuracy of finite element solution; that is to say, one substantially higher than warranted by the physical approximations inherent in the formulation of the problem, as well as the accuracy achievable in practical microcircuits. The model problem studies are held to bear out this expectation.

Conclusions

The method set out in this paper permits admittance matrices to be computed for arbitrary polygonal planar networks. Two distinct advantages over previously existing techniques render the method attractive. The admittance matrices are in partial-fraction form and therefore not restricted to any one frequency of operation. Two boundary-value problems need to be solved initially, but no others are required subsequently, regardless of frequency or nature of excitation. The boundary-value problem solution itself is accomplished by existing finite element programs, so that a major portion of the required programming work has already been accomplished.

Suitable programs for admittance matrix production have been written and tested on simple model problems for which analytic solutions exist. There appears to be reason to believe that, as a result of the formulation employed, even relatively crude finite element models suffice to produce admittance matrices of an accuracy compatible with the physical approximations inherent in the problem.

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